MPZ3132-Engineering Mathematics IB

Academic Year 2014/2015

Answer sheet for Assignment NO.01

1. (a) i. $f(x) = \cos x$

$f(x+2\pi) = cos(x+2\pi)$

$= cosxcos2\pi - sinxsin2\pi$

$= cosx$

$= f(x)$

Here $cosx$ is periodic of period 2π.

ii. $f(x) = \sin 4x$

$f(x+2/\pi) = sin(4(x+\pi/2))$

$= sin(4x+2\pi)$

$= sin4xcos2\pi + cos4xsin2\pi$

$= sin4x$

$= f(x)$

Here $sin4x$ is periodic of period $\pi/2$.

iii. $f(x) = \sin 3x$

$f(x+(2\pi)/3) = sin(3(x+2\pi/3))$

$= sin(3x+2\pi)$

$= sin3xcos2\pi + cos3xsin2\pi$

$= sin3x$

$= f(x)$

Here $sin3x$ is periodic of period $2\pi/3$.

(b) $f(x) = \begin{cases} 0, & \text{if } -\pi \le x \le 0; \\ 1, & \text{if } 0 < x \le \pi. \end{cases}$ and $f(x+2\pi) = f(x)$.

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0} dx + \int_{0}^{\pi} 1dx = \frac{1}{\pi} \times \pi = 1$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \cos nx dx = \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}[0-0] = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin nx dx = \frac{-1}{\pi}\left[\frac{\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{-1}{\pi}[\cos(n\pi) - 1] = \frac{1-\cos(n\pi)}{n\pi}$$

$$b_n = \frac{1-(-1)^n}{n\pi}$$

The Fourier series of function $f(x)$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$.

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n\pi} \sin nx.$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sin x + \frac{2}{3\pi}\sin 3x + \frac{2}{5\pi}\sin 5x + \dots$$

$$f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=0}^{\infty} \frac{\sin(2k+1)x}{(2k+1)}.$$

(c) $f(x) = e^{-t/2}$ $0 \le t \le \pi$.

$$a_0 = \frac{1}{\pi}\int_0^{\pi} e^{-t/2}dt = \frac{-2}{\pi}e^{-t/2}\Big|_0^{\pi} = \frac{-2}{\pi}[e^{-\pi/2} - e^0] = \frac{-2}{\pi}[0.2079 - 1]$$

$$a_0 = 0.504$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t)\cos(\frac{n\pi t}{L})dt$$

$$a_n = \frac{2}{L}\int_{0}^{L} f(t)\cos(\frac{n\pi t}{L})dt$$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} f(t)\cos 2nt dt = \frac{2}{\pi}\int_{0}^{\pi} e^{-t/2}\cos 2nt dt$$

$$= \frac{2}{\pi}\left[\left[e^{-t/2}\sin\frac{2nt}{2n}\right]_0^{\pi} + \frac{1}{4n}\int_0^{\pi} e^{-t/2}\sin 2nt dt\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{1}{4n}\int_0^{\pi} e^{-t/2}\sin 2nt dt\right]$$

$$= \frac{2}{\pi}\left[-e^{-t/2}\frac{\cos 2nt}{8n^2}\right]_0^{\pi} + \frac{2}{\pi}\left(-\frac{1}{16n^2}\int_0^{\pi} e^{-t/2}\cos 2nt dt\right)$$

$$a_n\left(1+\frac{1}{16n^2}\right)=\frac{-1}{4n^2\pi}(e^{-\pi/2}-1)$$
$$a_n\left(\frac{16n^2+1}{16n^2}\right)=\frac{-1}{4n^2\pi}(0.2079-1)$$
$$a_n=\left(\frac{16n^2}{16n^2+1}\right)\frac{1}{8n^2}\frac{-2}{\pi}(0.2079-1)$$
$$a_n=\frac{2(0.504)}{16n^2+1}$$
$$b_n=\frac{2}{\pi}\int_0^\pi f(t)\sin 2ntdt=\frac{2}{\pi}\int_0^\pi e^{-t/2}\sin 2ntdt$$
$$=\frac{2}{\pi}\left[\left[-\frac{1}{2n}e^{-t/2}\cos 2nt\right]_0^\pi-\frac{1}{4n}\int_0^\pi e^{-t/2}\cos 2ntdt\right]$$
$$=\frac{2}{\pi}\left[\left[-\frac{1}{2n}e^{-\pi/2}-1\right]-\frac{1}{4n}\frac{2}{\pi}\int_0^\pi e^{-t/2}\cos 2ntdt\right]$$
$$=\frac{1}{2n}(0.504)-\frac{2}{4n}\frac{(0.504)}{(16n^2+1)}$$
$$=\frac{(0.504)}{2n}\frac{(16n^2+1-1)}{(16n^2+1)}$$
$$=\frac{8n}{(16n^2+1)}(0.504)$$
$$\therefore f(t)=0.504\left[1+\sum_{n=1}^{\infty}\left[\frac{2\cos(2nt)}{(1+16n^2)}+\frac{8n\sin(2nt)}{1+16n^2}\right]\right]$$

(d) Fourier Sine expansion

$$a_0=a_n=0$$
$$b_n=\int_0^2 f(x)\sin\frac{n\pi x}{2}dx$$
$$b_n=\int_0^1 x\sin\frac{n\pi x}{2}dx+\int_1^2(2-x)\sin\frac{n\pi x}{2}dx$$
$$=\left(\frac{-2x}{n\pi}\cos\frac{n\pi x}{2}+\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\Big|_0^1+\left(\frac{-2(2-x)}{n\pi}\cos\frac{n\pi x}{2}-\frac{4}{n^2\pi^2}\sin\frac{n\pi x}{2}\right)\Big|_1^2$$
$$b_n=\frac{8}{n^2\pi^2}\sin\frac{n\pi}{2}$$

When $n=$ even, $b_n=0$

When $n=$ odd, $b_n=\dfrac{8(-1)^k}{(2k+1)^2\pi^2}$

$$\therefore f(x)=\frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^2}\sin\frac{(2k+1)\pi x}{2}$$

When $x=1$ the function $f(1)$ is equal to 1.

$$1=\frac{8}{\pi^2}\sum_{k=0}^{\infty}\frac{(-1)^k(-1)^k}{(2k+1)^2}$$

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$$

2. (a)
$$\begin{aligned}
a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} -1dx + \int_{0}^{\pi} dx\right] \\
&= \frac{1}{\pi}\left[-x\Big|_{-\pi}^{0} + x\Big|_{0}^{\pi}\right] \\
&= \frac{1}{\pi}\Big[-\pi + \pi\Big] = 0 \\
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\cos nx dx + \int_{0}^{\pi} \cos nx dx\right] \\
&= \frac{1}{\pi}\left[\frac{-\sin nx}{n}\Big|_{-\pi}^{0} + \frac{\sin nx}{n}\Big|_{0}^{\pi}\right] = 0 \\
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\left[\int_{-\pi}^{0} -\sin nx dx + \int_{0}^{\pi} \sin nx dx\right] \\
&= \frac{1}{\pi}\left[\frac{\cos nx}{n}\Big|_{-\pi}^{0} + \frac{-\cos nx}{n}\Big|_{0}^{\pi}\right] \\
&= \frac{1}{n\pi}\Big[1 - (-1)^n - (-1)^n + 1\Big] = \frac{2}{n\pi}[1 - (-1)^n]
\end{aligned}$$

$$b_n \begin{cases} 0, & \text{; n is even} \\ \dfrac{4}{n\pi}, & \text{; n is odd} \end{cases}$$

Now Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin(2n-1)x$$

Parseval's Formular

$$2|a_0|^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)dx$$

$$\sum_{n=1}^{\infty}\left[\frac{4}{(2n-1)\pi}\right]^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} dx$$

$$\frac{16}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = 2$$

$$\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$$

(b) $f(x) = \pi^2 - x^2$

$f(-x) = \pi^2 - (-x^2) = \pi^2 - x^2 = f(x)$ So f is even.

$b_n = 0$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos nx dx = \frac{2}{\pi}\int_{0}^{\pi} (\pi^2 - x^2)\cos nx dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}(\pi^2 - x^2)\sin nx + \frac{-2x}{n^2}\cos nx + \frac{2}{n^3}\sin nx\right]_0^{\pi}$$

$$= \frac{-4}{n^2\pi}[\pi(-1)^n] = \frac{-4(-1)^n}{n^2}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_{0}^{\pi} (\pi^2 - x^2)dx$$

$$= \frac{2}{\pi}\left[\pi^2 x - \frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\pi^3 - \frac{x^3}{3}\right] = \frac{4\pi^2}{3}$$

$\therefore$ Fourier series is $f(x) = \dfrac{2\pi^2}{3} + \displaystyle\sum_{n=1}^{\infty} \frac{-4}{n^2}(-1)^n \cos nx$

(c) As $f(-x) = |-x| = |x| = f(x)$

$|x|$ is an even function.

$$|x| = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Where $a_0 = \dfrac{2}{\pi}\displaystyle\int_{0}^{\pi} |x| dx$

$$a_0 = \frac{2}{\pi}\int_{0}^{\pi} x dx (\because |x| = x \text{ when } o < x < \pi)$$

$= \pi$

$$a_n = \frac{2}{\pi}\int_{0}^{\pi} |x|\cos nx dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\cos nx dx$$

$$= \frac{2}{\pi}\left[\frac{1}{n}x\sin nx + \frac{1}{n^2}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$a_n = \begin{cases} 0, & \text{n is even ;} \\ \dfrac{-4}{n^2\pi}, & \text{n is odd.} \end{cases}$$

$$\therefore f(x) = |x| = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi} cosnx$$

$$f(x) = |x| = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{cosnx}{n^2}$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\right]$$

Putting $x=0$, we have

$$0=\frac{\pi}{2}-\frac{4}{\pi}\left[1+\frac{1}{3^2}+\frac{1}{5^2}+\dots\right]$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi}{2}\frac{4}{\pi}$$

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2}=\frac{\pi^2}{8}$$

3. (a) Refer Figure 1: 3.1

   (b) $a_0=\frac{1}{\pi}\int_0^{2\pi}f(x)dx=\frac{1}{\pi}\int_0^{\pi}f(x)dx+\frac{1}{\pi}\int_{\pi}^{2\pi}f(x)dx$

$$=\frac{1}{\pi}\left[\int_0^{\pi}xdx+\int_{\pi}^{2\pi}\pi dx\right]$$

$$=\frac{1}{\pi}\left[\frac{\pi^2}{2}+\pi^2\right]=\frac{3\pi}{2}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\cos nxdx$$

$$=\frac{1}{\pi}\left[\int_0^{\pi}x\cos nxdx+\int_{\pi}^{2\pi}\pi\cos nxdx\right]$$

$$=\frac{1}{\pi}\left[\left(\frac{x}{n}\sin nx+\frac{\cos nx}{n^2}\right)\Big|_0^{\pi}+\frac{\pi}{n}\sin nx\Big|_{\pi}^{2\pi}\right]$$

$$=\frac{1}{n^2\pi}[(-1)^n-1]$$

$$a_n=\begin{cases}0, & \text{; n is even}\\ \frac{-2}{n^2\pi}, & \text{; n is odd.}\end{cases}$$

$$b_n=\frac{1}{\pi}\int_0^{2\pi}f(x)\sin nxdx$$

$$=\frac{1}{\pi}\left[\int_0^{\pi}x\sin nxdx+\int_{\pi}^{2\pi}\pi\sin nxdx\right]$$

$$=\frac{1}{\pi}\left[\left(-\frac{x}{n}\cos nx+\frac{\sin nx}{n^2}\right)\Big|_0^{\pi}+\left(\frac{\pi}{n}\right)-(\cos nx)\Big|_{\pi}^{2\pi}\right]$$

$$=\frac{1}{\pi}\left[\frac{-\pi}{n}(-1)^n-\frac{\pi}{n}(1-(-1)^n)\right]$$

$$=-\frac{1}{n}[(-1)^n+1-(1)^n]=-\frac{1}{n}$$

we have

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$$

$$=\frac{3\pi}{4}+\sum_{n=0}^{\infty}\frac{-2}{(2n+1)^2\pi}\cos(2n+1)x+\sum_{n=1}^{\infty}\frac{-1}{n}\sin nx$$

$$= \frac{3\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1} + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \dots\right] - \left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right]$$

(c) $f(x) = \frac{3\pi}{4} - \frac{2}{\pi}\left[\frac{\cos x}{1} + \frac{\cos 3x}{9} + \frac{\cos 5x}{25} + \dots\right] - \left[\sin x + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \dots\right]$

i. Here we set the cos nx terms to zero.

Let $x = \frac{\pi}{2}$

$f(\frac{\pi}{2}) = \frac{\pi}{2}$

$\frac{\pi}{2} = \frac{3\pi}{4} - \frac{2}{\pi}(0) - \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right]$

$\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right] = \frac{3\pi}{4} - \frac{2}{\pi} = \frac{\pi}{4}$

$\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots\right] = \frac{\pi}{4}$

ii. Here we set the sinx terms to zero.

Let $x = \pi$

$f(\pi) = \pi$

$\therefore \pi = \frac{3\pi}{4} - \frac{2}{\pi}\left[\cos\pi + \frac{\cos 3\pi}{9} + \frac{\cos 5\pi}{25} + \dots\right]$

$\frac{\pi}{4} = \frac{2}{\pi}\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \dots\right]$

$\left[1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + ...\right] = \frac{\pi^2}{8}$

4. (a) $f(x) = f(a) + f'(a)(x-a) + f''(a)\dfrac{(x-a)}{2!} + \ldots + \dfrac{f^n(a)(x-a)^n}{n!}$

$f(x) = \dfrac{1-x^2}{(1-2x\cos\theta + x^2)}$

$(1-2x\cos\theta+x^2)f(x) = 1-x^2$

$(1-2x\cos\theta+x^2)f'(x) + (2x-2\cos\theta)f(x) = -2x \longrightarrow 1$

$(1-2x\cos\theta+x^2)f''(x) + (2x-2\cos\theta)f'(x) + (2x-2\cos\theta)f'(x) + 2f(x) = -2$

$(1-2x\cos\theta+x^2)f''(x) + 4(x-\cos\theta)f'(x) + 2f(x) = -2 \longrightarrow 2$

$(1-2x\cos\theta+x^2)f'''(x)+(2x-2\cos\theta)f''(x)+4(x-\cos\theta)f''(x)+4f'(x)+2f'(x) = 0$

$(1-2x\cos\theta+x^2)f'''(x) + 6(x-\cos\theta)f''(x) + 6f'(x) = 0 \longrightarrow 3$

$(1-2x\cos\theta+x^2)f'^v(x)+(2x-2\cos\theta)f'''(x)+6(x-\cos\theta)f'''(x)+6f''(x)+6f''(x) =$

0

$(1-2x\cos\theta+x^2)f'^v(x) + 8(x-\cos\theta)f'''(x) + 12f''(x) = 0 \longrightarrow 4$

$(1-2x\cos\theta+x^2)f^v(x) + (2x-2\cos\theta)f'^v(x) + 8(x-\cos\theta)f'^v(x) + 8f'''(x) +$

$12f'''(x) = 0$

$(1-2x\cos\theta+x^2)f^v(x) + 10(x-\cos\theta)f'^v(x) + 20f'''(x) = 0 \longrightarrow 5$

$(1+x^2-2x\cos\theta)\dfrac{df^5(x)}{dx^5} + 10(x-\cos\theta)\dfrac{df^4(x)}{dx^4} + 20\dfrac{df^3(x)}{dx^3} = 0$

$f(0) = 1$

$1 \longrightarrow f'(0) - 2\cos\theta f(0) = 0$

$f'(0) = 2\cos\theta$

$2 \longrightarrow f''(0) - 4\cos\theta f'(0) + 2f(0) = -2$

$f''(0) - 4\cos\theta(2\cos\theta) + 2 = -2$

$f''(0) = -4 + 8\cos^2\theta = 4(2\cos^2\theta - 1) = 4\cos 2\theta$

$3 \longrightarrow f'''(0) - 6\cos\theta f''(0) + 6f'(0) = 0$

$f'''(0) = 6\cos\theta(4\cos 2\theta) - 12\cos\theta$

$f'''(0) = 6\cos\theta(8\cos^2\theta - 4) - 12\cos\theta = 48\cos^3\theta - 36\cos\theta$

$f'''(0) = 12(4\cos^3\theta - 3\cos\theta) = 12\cos 3\theta$

$4 \longrightarrow f'^v(0) - 8\cos\theta f'''(0) + 12f''(0) = 0$

$f'^v(0) = 8\cos\theta(48\cos^3\theta - 36\cos\theta) - 12(8\cos^2\theta - 4)$

$f'^v(0) = 384\cos^4\theta - 288\cos^2\theta - 96\cos^2\theta + 48$

$f'^v(0) = 384\cos^4\theta - 384\cos^2\theta + 48$

$$\left[\cos 2x = 2\cos^2 x - 1\right.$$
$$\cos 4x = \cos(2x + 2x)$$
$$= \cos^2 2x - sin^2 2x$$
$$= [2\cos^2 x - 1]^2 - [2\sin x \cos x]^2$$
$$= 4\cos^4 x - 4\cos^2 x + 1 - [4\cos^2 x(1 - \cos^2 x)]$$
$$= 4\cos^4 x - 4\cos^2 x + 1 - 4\cos^2 x + 4\cos^4 x$$
$$\left.\cos 4x = 8\cos^4 x - 8\cos^2 x + 1\right]$$
$$\therefore f'^{v}(0) = 48(8\cos^4\theta - 8\cos^2\theta + 1) = 48\cos 4\theta$$
Simillarly $f^{v}(0) = 240\cos 5\theta$
$$\therefore f(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f'^{v}(0)x^4}{4!} + \frac{f^{v}(0)x^5}{5!}$$
$$= 1 + 2\cos\theta x + 2\cos 2\theta x^2 + 2\cos 3\theta x^3 + 2\cos 4\theta x^4 + 2\cos 5\theta x^5$$
$$= 1 + \sum_{r=1}^{5} 2\cos(r\theta)x^r.$$

(b) $f(x) = \dfrac{1}{1-x} - 1$
$$f(0) = 0$$
$$f'(x) = \frac{1}{(1-x)^2} \Rightarrow f'(0) = 1$$
$$f''(x) = \frac{2}{(1-x)^3} \Rightarrow f''(0) = 2$$
$$f'''(x) = \frac{6}{(1-x)^3} \Rightarrow f'''(0) = 6$$
Linear approxiamation:
$$f(x) \simeq f(0) + xf'(0)$$
$$\simeq 0 + x = x$$
Quadratic approxiamation:
$$f(x) \simeq f(0) + xf'(0) + f''(0)\frac{x^2}{2!}$$
$$\simeq 0 + x + \frac{2}{2}x^2 = x + x^2$$
Cubic approxiamation:
$$f(x) \simeq f(0) + xf'(0) + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!}$$
$$\simeq 0 + x + x^2 + \frac{6}{3!}x^3$$
$$\simeq x + x^2 + x^3$$

(c) i. $f_1(x) = \ln(2 - x)$

$f_1(0) = \ln 2$

$f_1'(x) = \dfrac{-1}{(2-x)} \Rightarrow f_1'(0) = \dfrac{-1}{2}$

$(2-x)f_1'(x) = -1$

$(2-x)f_1''(x) - f_1'(x) = 0 \Rightarrow 2f_1''(0) = \dfrac{-1}{2} = f_1''(0) = \dfrac{-1}{4}$

$(2-x)f_1'''(x) - f_1''(x) - f_1''(x) = 0$

$(2-x)f_1'''(x) - 2f_1''(x) = 0 \Rightarrow 2f_1'''(0) = 2f_1''(0) = 2\dfrac{-1}{4} = \dfrac{-1}{2} = f_1'''(0) = \dfrac{-1}{4}$

$(2-x)f_1'^{v}(x) - f_1'''(x) - 2f_1'''(x) = 0$

$(2-x)f_1'^{v}(x) - 3f_1'''(x) = 0 \Rightarrow 2f_1'^{v}(x) = 3\dfrac{-1}{4} = f_1'^{v}(0) = \dfrac{-3}{8}$

$(2-x)f_1^{v}(x) - f_1'^{v}(x) - 3f_1'^{v}(x) = 0$

$(2-x)f_1^{v}(x) - 4f_1'^{v}(x) = 0 \Rightarrow 2f_1^{v}(0) = 4\dfrac{-3}{8} = f_1^{v}(0) = \dfrac{-3}{4}$

$\therefore$ Taylor Polynomial is

$$\begin{aligned}
f(x) &= f(0) + f'(0)x + f''(0)\frac{x^2}{2} + f'''(0)\frac{x^3}{3!} + f'^{v}(0)\frac{x^4}{4!} + f^{v}(0)\frac{x^5}{5!} \\
&= \ln 2 - \frac{1}{2}x - \frac{1}{4}\frac{x^2}{2} - \frac{1}{4}\frac{x^3}{6} - \frac{3}{8}\frac{x^4}{24} - \frac{3}{4}\frac{x^5}{120} \\
&= \ln 2 - \frac{x}{2} - \frac{x^2}{8} - \frac{x^3}{24} - \frac{x^4}{64} - \frac{x^5}{160}
\end{aligned}$$

ii. $f_2(x) = \dfrac{1}{(1-x)^2} \qquad \Rightarrow f_2(0) = 1$

$(1-x)^2 f_2'(x) + 2(1-x)(-1)f_2(x) = 0$

$(1-x)^2 f_2'(x) - 2(1-x)f_2(x) = 0 \Rightarrow f_2'(0) = 2$

$(1-x)^2 f_2''(x) - 2(1-x)f_2'(x) - 2(1-x)f_2'(x) + 2f_2(x) = 0$

$(1-x)^2 f_2''(x) - 4(1-x)f_2'(x) + 2f_2(x) = 0$

When $x = 0$

$f_2''(0) - 4(2) + 2(1) = 0 \Rightarrow f_2''(0) = 6$

$(1-x)^2 f_2'''(x) - 2(1-x)f_2''(x) - 4(1-x)f_2''(x) + 4f_2'(x) + 2f_2'(x) = 0$

$(1-x)^2 f_2'''(x) - 6(1-x)f_2''(x) + 6f_2'(x) = 0$

When $x = 0$

$f_2'''(0) - 6(6) + 6(2) = 0 \Rightarrow f_2'''(0) = 24$

$(1-x)^2 f_2'^{v}(x) - 2(1-x)f_2'''(x) - 6(1-x)f_2'''(x) + 6f_2''(x) + 6f_2''(x) = 0$

$(1-x)^2 f_2'^{v}(x) - 8(1-x)f_2'''(x) + 12f_2''(x) = 0$

When $x = 0$

$f_2^{\prime v}(0) - 8(24) + 12(6) = 0 \Rightarrow f_2^{\prime v}(0) = 120$

$(1-x)^2 f_2^{v}(x) - 2(1-x)f_2^{\prime v}(x) - 8(1-x)f_2^{\prime v}(x) + 8f_2^{\prime\prime\prime}(x) + 12f_2^{\prime\prime\prime}(x) = 0$

$(1-x)^2 f_2^{v}(x) - 10(1-x)f_2^{\prime v}(x) + 20f_2^{\prime\prime\prime}(x) = 0$

When $x = 0$

$f_2^{v}(0) - 10(120) + 20(24) = 0$

$f_2^{v}(0) = 720$

$\therefore$ Taylor Series is

$$f_2(x) = f_2(0) + f_2'(0)x + f_2''(0)\frac{x^2}{2} + f_2'''(0)\frac{x^3}{3!} + f_2^{\prime v}(0)\frac{x^4}{4!} + f_2^{v}(0)\frac{x^5}{5!}$$
$$= 1 + 2x + \frac{6}{2!}x^2 + \frac{24}{3!}x^3 + \frac{120}{4!}x^4 + \frac{720}{5!}x^5$$
$$= 1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5$$

5. (a) i. $f(x) = \ln(\cos x)$

$$f'(x) = \frac{1}{\cos x}(-\sin x) = (-\tan x)$$
$$f''(x) = -\sec^2 x = -1 - \tan^2 x$$
$$f''(x) = -1 - \left[\frac{df}{dx}\right]^2$$
$$\frac{d^3 f}{dx^3} = -2\frac{df}{dx}\frac{d^2 f}{dx^2}$$
$$\frac{d^4 f}{dx^4} = -2\frac{df}{dx}\frac{d^3 f}{dx^3} - 2\frac{d^2 f}{dx^2}\frac{d^2 f}{dx^2}$$
$$\frac{d^4 f}{dx^4} = -2\frac{df}{dx}\frac{d^3 f}{dx^3} - \left[2\frac{d^2 f}{dx^2}\right]^2$$

$f(0) = \ln(1) = 0$

$f'(0) = \tan(0) = 0$

$f''(0) = -1$

$f'''(0) = -2(0)(-1) = 0$

$f^{\prime v}(0) = -2(0)(0) - 2(1) = -2$

$$f(x) = -\frac{x^2}{2} - \frac{x^4}{12} + \ldots\ldots\ldots\ldots$$
$$\ln(\cos x) = -\frac{x^2}{2} - \frac{x^4}{12} + \ldots\ldots\ldots\ldots$$

$$\frac{\ln(\cos x)}{x^2} = -\frac{1}{2} - \frac{x^2}{12} + \text{.............}$$

$$\lim_{x\to 0}\frac{\ln(\cos x)}{x^2} = \lim_{x\to 0}\left[-\frac{1}{2} - \frac{x^2}{12} + \text{.............}\right] = -\frac{1}{2}$$

ii. $f(x) = e^x$

$$f(x) = e^x = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \text{..........}$$

$$f(x) = e^x \implies f(0) = 1$$

$$f'(x) = e^x \implies f'(0) = 1 \vdots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$$

$$= \sum_{r=0}^{\infty}\frac{x^r}{r!}$$

(b) $f(x) = \sin x$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^4(x) = \sin x$$

$$f^5(x) = \cos x$$

$$f^6(x) = -\sin x$$

$$f^7(x) = -\cos x$$

$$f^{2n}(x) = (-1)^n \sin x$$

$$f^{2n+1}(x) = (-1)^n \cos x$$

$$f^{2n}(0) = 0 f^{2n+1}(0) = (-1)^n$$

$$f(x) = f(0) + f'(0)x + f''(0)\frac{x^2}{2!} + f'''(0)\frac{x^3}{3!} + \dots$$

$$f(x) = 0 + x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\therefore \sin x = \sum_{n=0}^{\infty}\frac{(-1)^n(x)^{2n+1}}{(2n+1)!}$$

$$f(x) = \cos x$$

$$f'(x) = -\sin x$$

$$f''(x) = -\cos x$$

$f^3(x) = \sin x$

$f^4(x) = \cos x$

$f^5(x) = -\sin x$

$f^6(x) = -\cos x$

$f^7(x) = \sin x$

$f^{2n}(x) = (-1)^n \cos x$

$f^{2n+1}(x) = (-1)^n \sin x$

$\therefore \cos x = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2r)!}$

$\sin^2 x = \frac{1 - \cos 2x}{2}$

$= \frac{1}{2} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n 2x^n}{(2n)!}$

$= \frac{1}{2} - \frac{1}{2} \sum_{r=0}^{\infty} \frac{(-1)^n 2^n x^n}{(2n)!}$

$\cos^2 x = \frac{1}{2} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{(-1)^n (x)^n 2^{2n+1}}{(2n+1)!}$

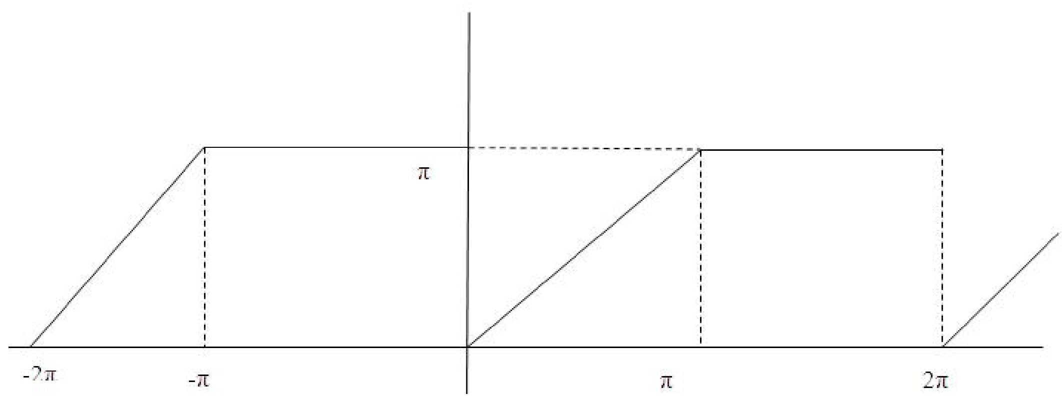

Figure 1: 3.1